International Journal of Mathematics, Statistics and Operations Research
Vol. 3; Number 2; 2023, pp. 281-300
https://doi.org/10.47509/IJMSOR.2023.v03i02.06

ARF INDIA
Academic Open Access Publishing
www.arfjournals.com

Sums and Generating Functions of Squares of Generalized Tribonacci Polynomials: Closed Formulas of $\sum_{k=0}^{n} z^k W_k^2$ and $\sum_{n=0}^{\infty} W_n^2 z^n$

Yüksel Soykan

Department of Mathematics,

Faculty of Science,

Zonguldak Bülent Ecevit University,

67100, Zonguldak, Turkey

e-mail: yuksel_soykan@hotmail.com

Abstract. In this paper, the closed forms of the sum formulas $\sum_{k=0}^{n} z^k W_k^2$, $\sum_{k=0}^{n} z^k W_{k+1} W_k$ and $\sum_{k=0}^{n} z^k W_{k+2} W_k$ for the generalized Tribonacci polynomials are presented. We also present the closed forms of formulas of generating functions $\sum_{n=0}^{\infty} W_n^2 z^n$, $\sum_{n=0}^{\infty} W_{n+1} W_n z^n$ and $\sum_{n=0}^{\infty} W_{n+2} W_n z^n$.

2020 Mathematics Subject Classification. 11B37, 11B39, 11B83.

Keywords. Sums of squares, third order recurrence, Tribonacci polynomials, Padovan polynomials, Perrin polynomials, Narayana polynomials, Tribonacci numbers, Padovan numbers, Perrin numbers, Narayana numbers.

1. Introduction

The generalized Tribonacci polynomials (or generalized $(r(x), s(x), t(x))-$Tribonacci polynomials or x-Tribonacci numbers or generalized $(r(x), s(x), t(x))$-polynomials or 3-step Fibonacci polynomials)

$$\{W_n(W_0(x), W_1(x), W_2(x); r(x), s(x), t(x))\}_{n\geq 0}$$

(or $\{W_n(x)\}_{n\geq 0}$ or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n(x) = r(x)W_{n-1}(x)+s(x)W_{n-2}(x)+t(x)W_{n-3}(x), \qquad W_0(x) = a(x), W_1(x) = b(x), W_2(x) = c(x), \quad n \geq 3 \tag{1.1}$$

where $W_0(x), W_1(x), W_2(x)$ are arbitrary complex (or real) polynomials with real coefficients and $r(x)$, $s(x)$ and $t(x)$ are polynomials with real coefficients and $t(x) \neq 0$.

Special cases of this sequence has been studied by many authors. For some references on special cases of generalized Tribonacci polynomials, see for example [1,2,3,4,5].

Article History
Received : 04 October 2022; Revised : 22 October 2023; Accepted : 06 November 2023; Published : 26 December 2023

To cite this paper
Yüksel Soykan (2023). Sums and Generating Functions of Squares of Generalized Tribonacci Polynomials: Closed Formulas of $\Sigma_{k=0}^{n} z^k W_k^2$ and $\Sigma_{n=0}^{\infty} W_n^2 z^n$. *International Journal of Mathematics, Statistics and Operations Research.* 3(2), 281-300.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n}(x) = -\frac{s(x)}{t(x)}W_{-(n-1)}(x) - \frac{r(x)}{t(x)}W_{-(n-2)}(x) + \frac{1}{t(x)}W_{-(n-3)}(x)$$

for $n = 1, 2, 3, ...$ when $t(x) \neq 0$. Therefore, recurrence (1.1) holds for all integers n. Note that for $n \geq 1$, $W_{-n}(x)$ need not to be a polynomial in the ordinary sense.

Binet's formula of generalized Tribonacci polynomials, as $\{W_n\}$ is a third-order recurrence sequence (difference equation), can be calculated using its characteristic equation which is given as

$$z^3 - r(x)z^2 - s(x)z - t(x) = 0. \tag{1.2}$$

The roots of characteristic equation of $\{W_n\}$ will be denoted as $\alpha(x) = \alpha(x, r, s, t), \beta(x) = \beta(x, r, s, t), \gamma(x) = \gamma(x, r, s, t)$.

Remark 1.1. *For the sake of simplicity throughout the rest of the paper, we use*

$$W_n, r, s, t, W_0, W_1, W_2, \alpha, \beta, \gamma,$$

instead of

$$W_n(x), r(x), s(x), t(x), W_0(x), W_1(x), W_2(x), \alpha(x), \beta(x), \gamma(x),$$

respectively, unless otherwise stated. For example, we write

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \qquad W_0 = a, W_1 = b, W_2 = c, \quad n \geq 3$$

for the equation (1.1).

Theorem 1.2. *[5, Theorem 6] Binet's formula of generalized Tribonacci polynomials is given as follows according to the roots of characteristic equation (1.2):*

(a): *(Three Distinct Roots Case: $\alpha \neq \beta \neq \gamma$)*

$$\begin{aligned} W_n &= \frac{W_2 - (\beta+\gamma)W_1 + \beta\gamma W_0}{(\alpha-\beta)(\alpha-\gamma)}\alpha^n + \frac{W_2 - (\alpha+\gamma)W_1 + \alpha\gamma W_0}{(\beta-\alpha)(\beta-\gamma)}\beta^n \\ &\quad + \frac{W_2 - (\alpha+\beta)W_1 + \alpha\beta W_0}{(\gamma-\alpha)(\gamma-\beta)}\gamma^n, \end{aligned}$$

i.e.,

$$\begin{aligned} W_n &= \frac{(\alpha W_2 + \alpha(-r+\alpha)W_1 + tW_0)}{r\alpha^2 + 2s\alpha + 3t}\alpha^n + \frac{(\beta W_2 + \beta(-r+\beta)W_1 + tW_0)}{r\beta^2 + 2s\beta + 3t}\beta^n \\ &\quad + \frac{(\gamma W_2 + \gamma(-r+\gamma)W_1 + tW_0)}{r\gamma^2 + 2s\gamma + 3t}\gamma^n. \end{aligned}$$

(b): *(Two Distinct Roots Case: $\alpha \neq \beta = \gamma$)*

$$W_n = \frac{W_2 - 2\beta W_1 + \beta^2 W_0}{(\beta - \alpha)^2}\alpha^n + (\frac{-W_2 + 2\beta W_1 - \alpha(2\beta - \alpha)W_0}{(\beta - \alpha)^2} + \frac{W_2 - (\beta + \alpha)W_1 + \beta\alpha W_0}{\beta(\beta - \alpha)}n)\beta^n$$

i.e.,

$$\begin{aligned} W_n &= \frac{4W_2 - 4(r-\alpha)W_1 + (r-\alpha)^2 W_0}{(r - 3\alpha)^2}\alpha^n \\ &\quad + \frac{1}{\beta(r - 3\beta)^2}((-\beta W_2 + 2\beta^2 W_1 + (2r\beta^2 + (r^2 + 8s)\beta + 8t)W_0) \\ &\quad + ((3\beta - r)W_2 - (r - 3\beta)(\beta - r)W_1 - (r\beta^2 + (r^2 + 6s)\beta + 6t)W_0)n)\beta^n. \end{aligned}$$

(c): *(Single Root Case: $\alpha = \beta = \gamma = \frac{r}{3}$)*

$$\begin{aligned} W_n &= \frac{1}{2}(2\alpha^2 W_0 + (-W_2 + 4W_1\alpha - 3W_0\alpha^2)n + (W_2 - 2W_1\alpha + W_0\alpha^2)n^2)\alpha^{n-2} \\ &= \frac{1}{2}(n(n-1)W_2 - 2n(n-2)\alpha W_1 + (n-1)(n-2)\alpha^2 W_0)\alpha^{n-2} \\ &= \frac{1}{18}(9n(n-1)W_2 - 6n(n-2)rW_1 + (n-1)(n-2)r^2 W_0)\left(\frac{r}{3}\right)^{n-2}. \end{aligned}$$

2. Sum Formulas

In this section, we present the closed forms of the sum formulas $\sum_{k=0}^{n} z^k W_k^2$, $\sum_{k=0}^{n} z^k W_{k+1}W_k$ and $\sum_{k=0}^{n} z^k W_{k+2}W_k$ for the generalized Tribonacci polynomials.

THEOREM 2.1. *Let z be a real or complex number. Then*

(a):

(i): *If* $\Gamma(z) = (-t^2z^3 + sz + rtz^2 + 1)(r^2z - s^2z^2 + t^2z^3 + 2sz + 2rtz^2 - 1) = -z^6t^4 + z^5t^2(s^2 - rt) + z^4t(r^2t - rs^2 - st) + z^3(r^3t - s^3 + 2t^2 + 4rst) + z^2(r^2s + s^2 + rt) + z(s + r^2) - 1 \neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_k^2 = \frac{\Theta_{1W}(z)}{\Gamma(z)} \tag{2.1}$$

where

$\Theta_{1W}(z) = -z^{n+6}\Theta_1 - z^{n+5}\Theta_2 - z^{n+4}\Theta_3 + z^{n+3}\Theta_4 + z^{n+2}\Theta_5 + z^{n+1}\Theta_6 + z^5\Theta_7 + z^4\Theta_8 + z^3\Theta_9 + z^2\Theta_{10} + z\Theta_{11} + \Theta_{12}$

$= -z^{n+6}t^2(W_{n+3}^2 + r^2W_{n+2}^2 + s^2W_{n+1}^2 + 2(-rW_{n+2}W_{n+3} - sW_{n+1}W_{n+3} + rsW_{n+1}W_{n+2})) - z^{n+5}t(rW_{n+3}^2 + (r^3 + 2rs + t)W_{n+2}^2 + r(rt - s^2)W_{n+1}^2 + 2(-(s + r^2)W_{n+3} + (s^2 - tr)W_{n+1})W_{n+2}) - z^{n+4}(sW_{n+3}^2 + r(t + rs)W_{n+2}^2 + (r^3t - s^3 + t^2 + 4rst)W_{n+1}^2 + 2(-rsW_{n+2}W_{n+3} - stW_{n+1}W_{n+2} - rtW_{n+1}W_{n+3}))$

$+z^{n+3}(W_{n+3}^2 - (s + r^2)W_{n+2}^2 - (r^2s + rt + s^2)W_{n+1}^2) + z^{n+2}(W_{n+2}^2 - (s + r^2)W_{n+1}^2) + z^{n+1}W_{n+1}^2 + z^5t^2(-W_2 + rW_1 + sW_0)^2 + z^4t(rW_2^2 + (t + 2rs + r^3)W_1^2 + r(rt - s^2)W_0^2 - 2(s + r^2)W_1W_2 - 2(rt - s^2)W_0W_1) + z^3(sW_2^2 + r(t + rs)W_1^2 + (r^3t - s^3 + t^2 + 4rst)W_0^2 - 2rsW_1W_2 - 2rtW_0W_2 - 2stW_0W_1)$

$+z^2(-W_2^2 + (r^2 + s)W_1^2 + s(s + r^2)W_0^2 + rtW_0^2) + z(-W_1^2 + (r^2 + s)W_0^2) - W_0^2$

(ii): *If* $\Gamma(z)=(-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1)=(z-a_1)f(z)=0$ *for some* $a_1\in\mathbb{C}$ *and a function* f *in* z *with* $f(a_1)\neq 0$ *then, for* $z=a_1$*, we get*

$$\sum_{k=0}^{n} z^k W_k^2=\frac{\frac{d}{dz}\Theta_{1W}(z)}{\frac{d}{dz}\Gamma(z)} \tag{2.2}$$

where

$\frac{d}{dz}\Theta_{1W}(z)=-(n+6)z^{n+5}t^2(W_{n+3}^2+r^2W_{n+2}^2+s^2W_{n+1}^2+2(-rW_{n+2}W_{n+3}-sW_{n+1}W_{n+3}+rsW_{n+1}W_{n+2}))-(n+5)z^{n+4}t(rW_{n+3}^2+(r^3+2rs+t)W_{n+2}^2+r(rt-s^2)W_{n+1}^2+2(-(s+r^2)W_{n+3}+(s^2-tr)W_{n+1})W_{n+2})-(n+4)z^{n+3}(sW_{n+3}^2+r(t+rs)W_{n+2}^2+(r^3t-s^3+t^2+4rst)W_{n+1}^2+2(-rsW_{n+2}W_{n+3}-stW_{n+1}W_{n+2}-rtW_{n+1}W_{n+3}))$
$+(n+3)z^{n+2}(W_{n+3}^2-(s+r^2)W_{n+2}^2-(r^2s+rt+s^2)W_{n+1}^2)+(n+2)z^{n+1}(W_{n+2}^2-(s+r^2)W_{n+1}^2)+(n+1)z^nW_{n+1}^2+5z^4t^2(-W_2+rW_1+sW_0)^2+4z^3t(rW_2^2+(t+2rs+r^3)W_1^2+r(rt-s^2)W_0^2-2(s+r^2)W_1W_2-2(rt-s^2)W_0W_1)+3z^2(sW_2^2+r(t+rs)W_1^2+(r^3t-s^3+t^2+4rst)W_0^2-2rsW_1W_2-2rtW_0W_2-2stW_0W_1)$
$+2z(-W_2^2+(r^2+s)W_1^2+s(s+r^2)W_0^2+rtW_0^2)+(-W_1^2+(r^2+s)W_0^2)$

and

$\frac{d}{dz}\Gamma(z)=-6z^5t^4+5z^4t^2(s^2-rt)+4z^3t(r^2t-rs^2-st)+3z^2(r^3t-s^3+2t^2+4rst)+2z(r^2s+s^2+rt)+(s+r^2)$

(iii): *If* $\Gamma(z)=(-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1)=(z-a_1)^2f(z)=0$ *for some* $a_1\in\mathbb{C}$ *and a function* f *in* z *with* $f(a_1)\neq 0$ *then, for* $z=a_1$*, we get*

$$\sum_{k=0}^{n} z^k W_k^2=\frac{\frac{d^2}{dz^2}\Theta_{1W}(z)}{\frac{d^2}{dz^2}\Gamma(z)} \tag{2.3}$$

where

$\frac{d^2}{dz^2}\Theta_{1W}(z)=-(n+6)(n+5)z^{n+4}t^2(W_{n+3}^2+r^2W_{n+2}^2+s^2W_{n+1}^2+2(-rW_{n+2}W_{n+3}-sW_{n+1}W_{n+3}+rsW_{n+1}W_{n+2}))-(n+5)(n+4)z^{n+3}t(rW_{n+3}^2+(r^3+2rs+t)W_{n+2}^2+r(rt-s^2)W_{n+1}^2+2(-(s+r^2)W_{n+3}+(s^2-tr)W_{n+1})W_{n+2})$
$-(n+4)(n+3)z^{n+2}(sW_{n+3}^2+r(t+rs)W_{n+2}^2+(r^3t-s^3+t^2+4rst)W_{n+1}^2+2(-rsW_{n+2}W_{n+3}-stW_{n+1}W_{n+2}-rtW_{n+1}W_{n+3}))+(n+3)(n+2)z^{n+1}(W_{n+3}^2-(s+r^2)W_{n+2}^2-(r^2s+rt+s^2)W_{n+1}^2)+(n+2)(n+1)z^n(W_{n+2}^2-(s+r^2)W_{n+1}^2)+(n+1)nz^{n-1}W_{n+1}^2$
$+20z^3t^2(-W_2+rW_1+sW_0)^2+12z^2t(rW_2^2+(t+2rs+r^3)W_1^2+r(rt-s^2)W_0^2-2(s+r^2)W_1W_2-2(rt-s^2)W_0W_1)+6z(sW_2^2+r(t+rs)W_1^2+(r^3t-s^3+t^2+4rst)W_0^2-2rsW_1W_2-2rtW_0W_2-2stW_0W_1)+2(-W_2^2+(r^2+s)W_1^2+s(s+r^2)W_0^2+rtW_0^2)$

and

$\frac{d^2}{dz^2}\Gamma(z)=-30z^4t^4+20z^3t^2(s^2-rt)+12z^2t(r^2t-rs^2-st)+6z(r^3t-s^3+2t^2+4rst)+2(r^2s+s^2+rt)$

(iv): *If* $\Gamma(z) = (-t^2z^3 + sz + rtz^2 + 1)(r^2z - s^2z^2 + t^2z^3 + 2sz + 2rtz^2 - 1) = (z - a_1)^3 f(z) = 0$ *for some* $a_1 \in \mathbb{C}$ *and a function* f *in* z *with* $f(a_1) \neq 0$ *then, for* $z = a_1$, *we get*

$$\sum_{k=0}^{n} z^k W_k^2 = \frac{\frac{d^3}{dz^3}\Theta_{1W}(z)}{\frac{d^3}{dz^3}\Gamma(z)} \tag{2.4}$$

$$= \frac{\frac{d^3}{dz^3}\Theta_{1W}(z)}{-120z^3t^4 + 60z^2t^2(s^2 - rt) + 24zt(r^2t - rs^2 - st) + 6(r^3t - s^3 + 2t^2 + 4rst)}$$

where

$\frac{d^3}{dz^3}\Theta_{1W}(z) = -(n+4)(n+5)(n+6)z^{n+3}t^2(W_{n+3}^2 + r^2W_{n+2}^2 + s^2W_{n+1}^2 + 2(-rW_{n+2}W_{n+3} - sW_{n+1}W_{n+3} + rsW_{n+1}W_{n+2})) - (n+3)(n+4)(n+5)z^{n+2}t(rW_{n+3}^2 + (r^3 + 2rs + t)W_{n+2}^2 + r(rt - s^2)W_{n+1}^2 + 2(-(s + r^2)W_{n+3} + (s^2 - tr)W_{n+1})W_{n+2})$
$-(n+2)(n+3)(n+4)z^{n+1}(sW_{n+3}^2 + r(t+rs)W_{n+2}^2 + (r^3t - s^3 + t^2 + 4rst)W_{n+1}^2 + 2(-rsW_{n+2}W_{n+3} - stW_{n+1}W_{n+2} - rtW_{n+1}W_{n+3})) + (n+1)(n+2)(n+3)z^n(W_{n+3}^2 - (s + r^2)W_{n+2}^2 - (r^2s + rt + s^2)W_{n+1}^2) + n(n+1)(n+2)z^{n-1}(W_{n+2}^2 - (s + r^2)W_{n+1}^2) + (n-1)n(n+1)z^{n-2}W_{n+1}^2$
$+60z^2t^2(-W_2 + rW_1 + sW_0)^2 + 24zt(rW_2^2 + (t + 2rs + r^3)W_1^2 + r(rt - s^2)W_0^2 - 2(s + r^2)W_1W_2 - 2(rt - s^2)W_0W_1) + 6(sW_2^2 + r(t + rs)W_1^2 + (r^3t - s^3 + t^2 + 4rst)W_0^2 - 2rsW_1W_2 - 2rtW_0W_2 - 2stW_0W_1)$

and

$\frac{d^3}{dz^3}\Gamma(z) = -120z^3t^4 + 60z^2t^2(s^2 - rt) + 24zt(r^2t - rs^2 - st) + 6(r^3t - s^3 + 2t^2 + 4rst)$

(v): *If* $\Gamma(z) = (-t^2z^3 + sz + rtz^2 + 1)(r^2z - s^2z^2 + t^2z^3 + 2sz + 2rtz^2 - 1) = (z - a_1)^4 f(z) = 0$ *for some* $a_1 \in \mathbb{C}$ *and a function* f *in* z *with* $f(a_1) \neq 0$ *then, for* $z = a_1$, *we get*

$$\sum_{k=0}^{n} z^k W_k^2 = \frac{\frac{d^4}{dz^4}\Theta_{1W}(z)}{\frac{d^4}{dz^4}\Gamma(z)} \tag{2.5}$$

$$= \frac{\frac{d^4}{dz^4}\Theta_{1W}(z)}{-360z^2t^4 + 120zt^2(s^2 - rt) + 24t(r^2t - rs^2 - st)}$$

where

$\frac{d^4}{dz^4}\Theta_{1W}(z) = -(n+3)(n+4)(n+5)(n+6)z^{n+2}t^2(W_{n+3}^2 + r^2W_{n+2}^2 + s^2W_{n+1}^2 + 2(-rW_{n+2}W_{n+3} - sW_{n+1}W_{n+3} + rsW_{n+1}W_{n+2})) - (n+2)(n+3)(n+4)(n+5)z^{n+1}t(rW_{n+3}^2 + (r^3 + 2rs + t)W_{n+2}^2 + r(rt - s^2)W_{n+1}^2 + 2(-(s + r^2)W_{n+3} + (s^2 - tr)W_{n+1})W_{n+2})$
$-(n+1)(n+2)(n+3)(n+4)z^n(sW_{n+3}^2 + r(t + rs)W_{n+2}^2 + (r^3t - s^3 + t^2 + 4rst)W_{n+1}^2 + 2(-rsW_{n+2}W_{n+3} - stW_{n+1}W_{n+2} - rtW_{n+1}W_{n+3})) + n(n+1)(n+2)(n+3)z^{n-1}(W_{n+3}^2 - (s + r^2)W_{n+2}^2 - (r^2s + rt + s^2)W_{n+1}^2) + (n-1)n(n+1)(n+2)z^{n-2}(W_{n+2}^2 - (s + r^2)W_{n+1}^2) + (n-2)(n-1)n(n+1)z^{n-3}W_{n+1}^2$

$+120zt^2(-W_2+rW_1+sW_0)^2+24t(rW_2^2+(t+2rs+r^3)W_1^2+r(rt-s^2)W_0^2-2(s+r^2)W_1W_2-2(rt-s^2)W_0W_1)$

and

$$\frac{d^4}{dz^4}\Gamma(z)=-360z^2t^4+120zt^2(s^2-rt)+24t(r^2t-rs^2-st)$$

(vi): *If* $\Gamma(z)=(-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1)=(z-a_1)^5f(z)=0$ *for some* $a_1\in\mathbb{C}$ *and a function* f *in* z *with* $f(a_1)\neq 0$ *then, for* $z=a_1$, *we get*

$$\sum_{k=0}^{n}z^kW_k^2 = \frac{\frac{d^5}{dz^5}\Theta_{1W}(z)}{\frac{d^5}{dz^5}\Gamma(z)} = \frac{\frac{d^5}{dz^5}\Theta_{1W}(z)}{-720zt^4+120t^2(s^2-rt)} \tag{2.6}$$

where

$\frac{d^5}{dz^5}\Theta_{1W}(z) = -(n+2)(n+3)(n+4)(n+5)(n+6)z^{n+1}t^2(W_{n+3}^2+r^2W_{n+2}^2+s^2W_{n+1}^2+2(-rW_{n+2}W_{n+3}-sW_{n+1}W_{n+3}+rsW_{n+1}W_{n+2}))-(n+1)(n+2)(n+3)(n+4)(n+5)z^nt(rW_{n+3}^2+(r^3+2rs+t)W_{n+2}^2+r(rt-s^2)W_{n+1}^2+2(-(s+r^2)W_{n+3}+(s^2-tr)W_{n+1})W_{n+2})$
$-n(n+1)(n+2)(n+3)(n+4)z^{n-1}(sW_{n+3}^2+r(t+rs)W_{n+2}^2+(r^3t-s^3+t^2+4rst)W_{n+1}^2+2(-rsW_{n+2}W_{n+3}-stW_{n+1}W_{n+2}-rtW_{n+1}W_{n+3}))+(n-1)n(n+1)(n+2)(n+3)z^{n-2}(W_{n+3}^2-(s+r^2)W_{n+2}^2-(r^2s+rt+s^2)W_{n+1}^2)+(n-2)(n-1)n(n+1)(n+2)z^{n-3}(W_{n+2}^2-(s+r^2)W_{n+1}^2)+(n-3)(n-2)(n-1)n(n+1)z^{n-4}W_{n+1}^2$
$+120t^2(-W_2+rW_1+sW_0)^2$

and

$$\frac{d^5}{dz^5}\Gamma(z)=-720zt^4+120t^2(s^2-rt)$$

(vii): *If* $\Gamma(z)=(-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1)=(z-a_1)^6=0$ *for some* $a_1\in\mathbb{C}$ *then, for* $z=a_1$, *we get*

$$\sum_{k=0}^{n}z^kW_k^2 = \frac{\frac{d^6}{dz^6}\Theta_{1W}(z)}{\frac{d^6}{dz^6}\Gamma(z)} = \frac{\frac{d^6}{dz^6}\Theta_{1W}(z)}{-720t^4} \tag{2.7}$$

where

$\frac{d^6}{dz^6}\Theta_{1W}(z)=-(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)z^nt^2(W_{n+3}^2+r^2W_{n+2}^2+s^2W_{n+1}^2+2(-rW_{n+2}W_{n+3}-sW_{n+1}W_{n+3}+rsW_{n+1}W_{n+2}))-n(n+1)(n+2)(n+3)(n+4)(n+5)z^{n-1}t(rW_{n+3}^2+(r^3+2rs+t)W_{n+2}^2+r(rt-s^2)W_{n+1}^2+2(-(s+r^2)W_{n+3}+(s^2-tr)W_{n+1})W_{n+2})$

$-(n-1)n(n+1)(n+2)(n+3)(n+4)z^{n-2}(sW_{n+3}^2+r(t+rs)W_{n+2}^2+(r^3t-s^3+t^2+4rst)W_{n+1}^2+2(-rsW_{n+2}W_{n+3}-stW_{n+1}W_{n+2}-rtW_{n+1}W_{n+3}))+(n-2)(n-1)n(n+1)(n+2)(n+3)z^{n-3}(W_{n+3}^2-(s+r^2)W_{n+2}^2-(r^2s+rt+s^2)W_{n+1}^2)$
$+(n-3)(n-2)(n-1)n(n+1)(n+2)z^{n-4}(W_{n+2}^2-(s+r^2)W_{n+1}^2)+(n-4)(n-3)(n-2)(n-1)n(n+1)z^{n-5}W_{n+1}^2$

and

$\dfrac{d^6}{dz^6}\Gamma(z)=-720t^4$

(b):

(i): *If* $\Gamma(z)=(-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1)\neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_{k+1}W_k=\frac{\Theta_{2W}(z)}{\Gamma(z)} \tag{2.8}$$

where

$\Theta_{2W}(z)=z^{n+6}\Theta_{13}+z^{n+5}\Theta_{14}+z^{n+4}\Theta_{15}+z^{n+3}\Theta_{16}+z^{n+2}\Theta_{17}+z^{n+1}\Theta_{18}+z^5\Theta_{19}+z^4\Theta_{20}+z^3\Theta_{21}+z^2\Theta_{22}+z\Theta_{23}+\Theta_{24}$

$=z^{n+6}t^3(-W_{n+3}+rW_{n+2}+sW_{n+1})W_{n+1}+z^{n+5}t(-W_{n+3}+rW_{n+2}+sW_{n+1})(-sW_{n+3}+tW_{n+2}+rsW_{n+2})+z^{n+4}(s(t+rs)W_{n+2}^2+rt^2W_{n+1}^2-s^2W_{n+2}W_{n+3}+r^2tW_{n+1}W_{n+3}+(-r^3t+s^3-t^2-2rst)W_{n+2}W_{n+1})+z^{n+3}(rW_{n+3}^2-r^2W_{n+2}W_{n+3}+tW_{n+1}W_{n+3}-(r^2s+rt+s^2)W_{n+2}W_{n+1})$
$+z^{n+2}(W_{n+3}-(s+r^2)W_{n+1})W_{n+2}+z^{n+1}W_{n+1}W_{n+2}+z^5t^3(W_2-rW_1-sW_0)W_0+z^4t(W_2-rW_1-sW_0)(-sW_2+(rs+t)W_1)+z^3(-s(t+rs)W_1^2-rt^2W_0^2+s^2W_1W_2-r^2tW_0W_2+(r^3t-s^3+t^2+2rst)W_0W_1)+z^2(-rW_2^2+r^2W_1W_2-tW_0W_2+(r^2s+rt+s^2)W_0W_1)+z(-W_2+(r^2+s)W_0)W_1-W_0W_1$

(ii): *If* $\Gamma(z)=(-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1)=(z-a_1)f(z)=0$ *for some* $a_1\in\mathbb{C}$ *and a function* f *in* z *with and* $f(a_1)\neq 0$ *then, for* $z=a_1$*, we get*

$$\sum_{k=0}^{n} z^k W_{k+1}W_k=\frac{\frac{d}{dz}\Theta_{2W}(z)}{\frac{d}{dz}\Gamma(z)} \tag{2.9}$$

where

$\dfrac{d}{dz}\Theta_{2W}(z)=(n+6)z^{n+5}t^3(-W_{n+3}+rW_{n+2}+sW_{n+1})W_{n+1}+(n+5)z^{n+4}t(-W_{n+3}+rW_{n+2}+sW_{n+1})(-sW_{n+3}+tW_{n+2}+rsW_{n+2})+(n+4)z^{n+3}(s(t+rs)W_{n+2}^2+rt^2W_{n+1}^2-s^2W_{n+2}W_{n+3}+r^2tW_{n+1}W_{n+3}+(-r^3t+s^3-t^2-2rst)W_{n+2}W_{n+1})+(n+3)z^{n+2}(rW_{n+3}^2-r^2W_{n+2}W_{n+3}+tW_{n+1}W_{n+3}-(r^2s+rt+s^2)W_{n+2}W_{n+1})$
$+(n+2)z^{n+1}(W_{n+3}-(s+r^2)W_{n+1})W_{n+2}+(n+1)z^nW_{n+1}W_{n+2}+5z^4t^3(W_2-rW_1-sW_0)W_0+4z^3t(W_2-rW_1-sW_0)(-sW_2+(rs+t)W_1)+3z^2(-s(t+rs)W_1^2-rt^2W_0^2+s^2W_1W_2-r^2tW_0W_2+(r^3t-s^3+t^2+2rst)W_0W_1)+2z(-rW_2^2+r^2W_1W_2-tW_0W_2+(r^2s+rt+s^2)W_0W_1)+(-W_2+(r^2+s)W_0)W_1$

and

$\frac{d}{dz}\Gamma(z) = -6z^5t^4 + 5z^4t^2(s^2 - rt) + 4z^3t(r^2t - rs^2 - st) + 3z^2(r^3t - s^3 + 2t^2 + 4rst) + 2z(r^2s + s^2 + rt) + (s + r^2)$

(iii): *If* $\Gamma(z) = (-t^2z^3 + sz + rtz^2 + 1)(r^2z - s^2z^2 + t^2z^3 + 2sz + 2rtz^2 - 1) = (z - a_1)^2 f(z) = 0$ *for some* $a_1 \in \mathbb{C}$ *and a function* f *in* z *with* $f(a_1) \neq 0$ *then, for* $z = a_1$*, we get*

$$\sum_{k=0}^{n} z^k W_{k+1}W_k = \frac{\frac{d^2}{dz^2}\Theta_{2W}(z)}{\frac{d^2}{dz^2}\Gamma(z)} \tag{2.10}$$

where

$\frac{d^2}{dz^2}\Theta_{2W}(z) = (n+5)(n+6)z^{n+4}t^3(-W_{n+3}+rW_{n+2}+sW_{n+1})W_{n+1}+(n+4)(n+5)z^{n+3}t(-W_{n+3}+rW_{n+2}+sW_{n+1})(-sW_{n+3}+tW_{n+2}+rsW_{n+2})+(n+3)(n+4)z^{n+2}(s(t+rs)W_{n+2}^2+rt^2W_{n+1}^2-s^2W_{n+2}W_{n+3}+r^2tW_{n+1}W_{n+3}+(-r^3t+s^3-t^2-2rst)W_{n+2}W_{n+1})+(n+2)(n+3)z^{n+1}(rW_{n+3}^2-r^2W_{n+2}W_{n+3}+tW_{n+1}W_{n+3}-(r^2s+rt+s^2)W_{n+2}W_{n+1})$
$+(n+1)(n+2)z^n(W_{n+3}-(s+r^2)W_{n+1})W_{n+2}+n(n+1)z^{n-1}W_{n+1}W_{n+2}+20z^3t^3(W_2-rW_1-sW_0)W_0+12z^2t(W_2-rW_1-sW_0)(-sW_2+(rs+t)W_1)+6z(-s(t+rs)W_1^2-rt^2W_0^2+s^2W_1W_2-r^2tW_0W_2+(r^3t-s^3+t^2+2rst)W_0W_1)+2(-rW_2^2+r^2W_1W_2-tW_0W_2+(r^2s+rt+s^2)W_0W_1)$

and

$\frac{d^2}{dz^2}\Gamma(z) = -30z^4t^4 + 20z^3t^2(s^2 - rt) + 12z^2t(r^2t - rs^2 - st) + 6z(r^3t - s^3 + 2t^2 + 4rst) + 2(r^2s + s^2 + rt)$

(iv): *If* $\Gamma(z) = (-t^2z^3 + sz + rtz^2 + 1)(r^2z - s^2z^2 + t^2z^3 + 2sz + 2rtz^2 - 1) = (z - a_1)^3 f(z) = 0$ *for some* $a_1 \in \mathbb{C}$ *and a function* f *in* z *with* $f(a_1) \neq 0$ *then, for* $z = a_1$*, we get*

$$\begin{aligned}\sum_{k=0}^{n} z^k W_{k+1}W_k &= \frac{\frac{d^3}{dz^3}\Theta_{2W}(z)}{\frac{d^3}{dz^3}\Gamma(z)} \qquad (2.11)\\ &= \frac{\frac{d^3}{dz^3}\Theta_{2W}(z)}{-120z^3t^4 + 60z^2t^2(s^2 - rt) + 24zt(r^2t - rs^2 - st) + 6(r^3t - s^3 + 2t^2 + 4rst)}\end{aligned}$$

where

$\frac{d^3}{dz^3}\Theta_{2W}(z) = (n+4)(n+5)(n+6)z^{n+3}t^3(-W_{n+3}+rW_{n+2}+sW_{n+1})W_{n+1}+(n+3)(n+4)(n+5)z^{n+2}t(-W_{n+3}+rW_{n+2}+sW_{n+1})(-sW_{n+3}+tW_{n+2}+rsW_{n+2})+(n+2)(n+3)(n+4)z^{n+1}(s(t+rs)W_{n+2}^2+rt^2W_{n+1}^2-s^2W_{n+2}W_{n+3}+r^2tW_{n+1}W_{n+3}+(-r^3t+s^3-t^2-2rst)W_{n+2}W_{n+1})$
$+(n+1)(n+2)(n+3)z^n(rW_{n+3}^2-r^2W_{n+2}W_{n+3}+tW_{n+1}W_{n+3}-(r^2s+rt+s^2)W_{n+2}W_{n+1})+n(n+1)(n+2)z^{n-1}(W_{n+3}-(s+r^2)W_{n+1})W_{n+2}+(n-1)n(n+1)z^{n-2}W_{n+1}W_{n+2}+60z^2t^3(W_2-rW_1-sW_0)W_0+24zt(W_2-rW_1-sW_0)(-sW_2+(rs+t)W_1)+6(-s(t+rs)W_1^2-rt^2W_0^2+s^2W_1W_2-r^2tW_0W_2+(r^3t-s^3+t^2+2rst)W_0W_1)$

and

$\frac{d^3}{dz^3}\Gamma(z) = -120z^3t^4 + 60z^2t^2(s^2 - rt) + 24zt(r^2t - rs^2 - st) + 6(r^3t - s^3 + 2t^2 + 4rst)$

(v): *If* $\Gamma(z) = (-t^2z^3 + sz + rtz^2 + 1)(r^2z - s^2z^2 + t^2z^3 + 2sz + 2rtz^2 - 1) = (z - a_1)^4 f(z) = 0$ *for some* $a_1 \in \mathbb{C}$ *and a function* f *in* z *with* $f(a_1) \neq 0$ *then, for* $z = a_1$, *we get*

$$\sum_{k=0}^{n} z^k W_{k+1}W_k = \frac{\frac{d^4}{dz^4}\Theta_{2W}(z)}{\frac{d^4}{dz^4}\Gamma(z)} = \frac{\frac{d^4}{dz^4}\Theta_{2W}(z)}{-360z^2t^4 + 120zt^2(s^2 - rt) + 24t(r^2t - rs^2 - st)} \tag{2.12}$$

where
$\frac{d^4}{dz^4}\Theta_{2W}(z) = (n+3)(n+4)(n+5)(n+6)z^{n+2}t^3(-W_{n+3} + rW_{n+2} + sW_{n+1})W_{n+1} + (n+2)(n+3)(n+4)(n+5)z^{n+1}t(-W_{n+3} + rW_{n+2} + sW_{n+1})(-sW_{n+3} + tW_{n+2} + rsW_{n+2}) + (n+1)(n+2)(n+3)(n+4)z^n(s(t+rs)W_{n+2}^2 + rt^2W_{n+1}^2 - s^2W_{n+2}W_{n+3} + r^2tW_{n+1}W_{n+3} + (-r^3t + s^3 - t^2 - 2rst)W_{n+2}W_{n+1})$
$+n(n+1)(n+2)(n+3)z^{n-1}(rW_{n+3}^2 - r^2W_{n+2}W_{n+3} + tW_{n+1}W_{n+3} - (r^2s + rt + s^2)W_{n+2}W_{n+1}) + (n-1)n(n+1)(n+2)z^{n-2}(W_{n+3} - (s+r^2)W_{n+1})W_{n+2} + (n-2)(n-1)n(n+1)z^{n-3}W_{n+1}W_{n+2} + 120zt^3(W_2 - rW_1 - sW_0)W_0 + 24t(W_2 - rW_1 - sW_0)(-sW_2 + (rs+t)W_1)$

and
$\frac{d^4}{dz^4}\Gamma(z) = -360z^2t^4 + 120zt^2(s^2 - rt) + 24t(r^2t - rs^2 - st)$

(vi): *If* $\Gamma(z) = (-t^2z^3 + sz + rtz^2 + 1)(r^2z - s^2z^2 + t^2z^3 + 2sz + 2rtz^2 - 1) = (z - a_1)^5 f(z) = 0$ *for some* $a_1 \in \mathbb{C}$ *and a function* f *in* z *with* $f(a_1) \neq 0$ *then, for* $z = a_1$, *we get*

$$\sum_{k=0}^{n} z^k W_{k+1}W_k = \frac{\frac{d^5}{dz^5}\Theta_{2W}(z)}{\frac{d^5}{dz^5}\Gamma(z)} = \frac{\frac{d^5}{dz^5}\Theta_{2W}(z)}{-720zt^4 + 120t^2(s^2 - rt)} \tag{2.13}$$

where
$\frac{d^5}{dz^5}\Theta_{2W}(z) = (n+2)(n+3)(n+4)(n+5)(n+6)z^{n+1}t^3(-W_{n+3} + rW_{n+2} + sW_{n+1})W_{n+1} + (n+1)(n+2)(n+3)(n+4)(n+5)z^n t(-W_{n+3} + rW_{n+2} + sW_{n+1})(-sW_{n+3} + tW_{n+2} + rsW_{n+2}) + n(n+1)(n+2)(n+3)(n+4)z^{n-1}(s(t+rs)W_{n+2}^2 + rt^2W_{n+1}^2 - s^2W_{n+2}W_{n+3} + r^2tW_{n+1}W_{n+3} + (-r^3t + s^3 - t^2 - 2rst)W_{n+2}W_{n+1})$
$+(n-1)n(n+1)(n+2)(n+3)z^{n-2}(rW_{n+3}^2 - r^2W_{n+2}W_{n+3} + tW_{n+1}W_{n+3} - (r^2s + rt + s^2)W_{n+2}W_{n+1}) + (n-2)(n-1)n(n+1)(n+2)z^{n-3}(W_{n+3} - (s+r^2)W_{n+1})W_{n+2} + (n-3)(n-2)(n-1)n(n+1)z^{n-4}W_{n+1}W_{n+2} + 120t^3(W_2 - rW_1 - sW_0)W_0$

and
$\frac{d^5}{dz^5}\Gamma(z) = -720zt^4 + 120t^2(s^2 - rt)$

(vii): *If* $\Gamma(z)=(-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1)=(z-a_1)^6=0$ *for some* $a_1\in\mathbb{C}$ *then, for* $z=a_1$*, we get*

$$\sum_{k=0}^{n} z^kW_{k+1}W_k = \frac{\frac{d^6}{dz^6}\Theta_{2W}(z)}{\frac{d^6}{dz^6}\Gamma(z)} = \frac{\frac{d^6}{dz^6}\Theta_{2W}(z)}{-720t^4} \tag{2.14}$$

where
$\frac{d^6}{dz^6}\Theta_{2W}(z)=(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)z^nt^3(-W_{n+3}+rW_{n+2}+sW_{n+1})W_{n+1}+n(n+1)(n+2)(n+3)(n+4)(n+5)z^{n-1}t(-W_{n+3}+rW_{n+2}+sW_{n+1})(-sW_{n+3}+tW_{n+2}+rsW_{n+2})+(n-1)n(n+1)(n+2)(n+3)(n+4)z^{n-2}(s(t+rs)W_{n+2}^2+rt^2W_{n+1}^2-s^2W_{n+2}W_{n+3}+r^2tW_{n+1}W_{n+3}+(-r^3t+s^3-t^2-2rst)W_{n+2}W_{n+1})$
$+(n-2)(n-1)n(n+1)(n+2)(n+3)z^{n-3}(rW_{n+3}^2-r^2W_{n+2}W_{n+3}+tW_{n+1}W_{n+3}-(r^2s+rt+s^2)W_{n+2}W_{n+1})+(n-3)(n-2)(n-1)n(n+1)(n+2)z^{n-4}(W_{n+3}-(s+r^2)W_{n+1})W_{n+2}+(n-4)(n-3)(n-2)(n-1)n(n+1)z^{n-5}W_{n+1}W_{n+2}$

and
$\frac{d^6}{dz^6}\Gamma(z)=-720t^4$

(c):

(i): *If* $\Gamma(z)=(-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1)\neq 0$ *then*

$$\sum_{k=0}^{n} z^kW_{k+2}W_k=\frac{\Theta_{3W}(z)}{\Gamma(z)} \tag{2.15}$$

where

$\Theta_{3W}(z)=z^{n+6}\Theta_{25}+z^{n+5}\Theta_{26}+z^{n+4}\Theta_{27}+z^{n+3}\Theta_{28}+z^{n+2}\Theta_{29}+z^{n+1}\Theta_{30}+z^5\Theta_{31}+z^4\Theta_{32}+z^3\Theta_{33}+z^2\Theta_{34}+z\Theta_{35}+\Theta_{36}=z^{n+6}t^3(-W_{n+3}+rW_{n+2}+sW_{n+1})W_{n+2}+z^{n+5}t(r(rt-s^2)W_{n+2}^2+t(rt-s^2)W_{n+1}^2+(s^2-rt)W_{n+2}W_{n+3}-(s^3+t^2)W_{n+1}W_{n+2}+stW_{n+1}W_{n+3})+z^{n+4}((rt-s^2)W_{n+3}^2+t^2(r^2+s)W_{n+1}^2+r(s^2-rt)W_{n+2}W_{n+3}+(s^3-2rst-t^2)W_{n+1}W_{n+3}+st(r^2+s)W_{n+2}W_{n+1})$
$+z^{n+3}((r^2+s)W_{n+3}^2-s(r^2+s)W_{n+2}^2+(t-r^3)W_{n+2}W_{n+3}-s(r^2+s)W_{n+3}W_{n+1}-t(r^2+s)W_{n+2}W_{n+1})+z^{n+2}(sW_{n+2}^2+rW_{n+2}W_{n+3}-(r^2+s)W_{n+1}W_{n+3}+tW_{n+1}W_{n+2})+z^{n+1}W_{n+1}W_{n+3}+z^5t^3(W_2-rW_1-sW_0)W_1+z^4t(r(s^2-rt)W_1^2+tW_0^2(s^2-rt)+(rt-s^2)W_1W_2-stW_0W_2+(s^3+t^2)W_0W_1)$
$+z^3((s^2-rt)W_2^2-t^2(r^2+s)W_0^2+r\left(rt-s^2\right)W_1W_2+(t^2-s^3+2rst)W_0W_2-st(r^2+s)W_0W_1)+z^2(-(r^2+s)W_2^2+s(r^2+s)W_1^2+(r^3-t)W_1W_2+s(r^2+s)W_0W_2+t(r^2+s)W_0W_1)+z(-sW_1^2-rW_1W_2+(r^2+s)W_0W_2-tW_0W_1)-W_0W_2$

(ii): *If* $\Gamma(z) = (-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1) = (z-a_1)f(z) = 0$ *for some* $a_1 \in \mathbb{C}$ *and a function* f *in* z *with* $f(a_1) \neq 0$ *then, for* $z = a_1$*, we get*

$$\sum_{k=0}^{n} z^k W_{k+2}W_k = \frac{\frac{d}{dz}\Theta_{3W}(z)}{\frac{d}{dz}\Gamma(z)} \tag{2.16}$$

where

$\frac{d}{dz}\Theta_{3W}(z) = (n+6)z^{n+5}t^3(-W_{n+3}+rW_{n+2}+sW_{n+1})W_{n+2}+(n+5)z^{n+4}t(r(rt-s^2)W_{n+2}^2+t(rt-s^2)W_{n+1}^2+(s^2-rt)W_{n+2}W_{n+3}-(s^3+t^2)W_{n+1}W_{n+2}+stW_{n+1}W_{n+3})+(n+4)z^{n+3}((rt-s^2)W_{n+3}^2+t^2(r^2+s)W_{n+1}^2+r(s^2-rt)W_{n+2}W_{n+3}+(s^3-2rst-t^2)W_{n+1}W_{n+3}+st(r^2+s)W_{n+2}W_{n+1})$

$+(n+3)z^{n+2}((r^2+s)W_{n+3}^2-s(r^2+s)W_{n+2}^2+(t-r^3)W_{n+2}W_{n+3}-s(r^2+s)W_{n+3}W_{n+1}-t(r^2+s)W_{n+2}W_{n+1})+(n+2)z^{n+1}(sW_{n+2}^2+rW_{n+2}W_{n+3}-(r^2+s)W_{n+1}W_{n+3}+tW_{n+1}W_{n+2})+(n+1)z^nW_{n+1}W_{n+3}+5z^4t^3(W_2-rW_1-sW_0)W_1+4z^3t(r(s^2-rt)W_1^2+tW_0^2(s^2-rt)+(rt-s^2)W_1W_2-stW_0W_2+(s^3+t^2)W_0W_1)$

$+3z^2((s^2-rt)W_2^2-t^2(r^2+s)W_0^2+r\left(rt-s^2\right)W_1W_2+(t^2-s^3+2rst)W_0W_2-st(r^2+s)W_0W_1)+2z(-(r^2+s)W_2^2+s(r^2+s)W_1^2+(r^3-t)W_1W_2+s(r^2+s)W_0W_2+t(r^2+s)W_0W_1)+(-sW_1^2-rW_1W_2+(r^2+s)W_0W_2-tW_0W_1)$

and

$\frac{d}{dz}\Gamma(z) = -6z^5t^4+5z^4t^2(s^2-rt)+4z^3t(r^2t-rs^2-st)+3z^2(r^3t-s^3+2t^2+4rst)+2z(r^2s+s^2+rt)+(s+r^2)$

(iii): *If* $\Gamma(z) = (-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1) = (z-a_1)^2f(z) = 0$ *for some* $a_1 \in \mathbb{C}$ *and a function* f *in* z *with* $f(a_1) \neq 0$ *then, for* $z = a_1$*, we get*

$$\sum_{k=0}^{n} z^k W_{k+2}W_k = \frac{\frac{d^2}{dz^2}\Theta_{3W}(z)}{\frac{d^2}{dz^2}\Gamma(z)} \tag{2.17}$$

where

$\frac{d^2}{dz^2}\Theta_{3W}(z) = (n+5)(n+6)z^{n+4}t^3(-W_{n+3}+rW_{n+2}+sW_{n+1})W_{n+2}+(n+4)(n+5)z^{n+3}t(r(rt-s^2)W_{n+2}^2+t(rt-s^2)W_{n+1}^2+(s^2-rt)W_{n+2}W_{n+3}-(s^3+t^2)W_{n+1}W_{n+2}+stW_{n+1}W_{n+3})+(n+3)(n+4)z^{n+2}((rt-s^2)W_{n+3}^2+t^2(r^2+s)W_{n+1}^2+r(s^2-rt)W_{n+2}W_{n+3}+(s^3-2rst-t^2)W_{n+1}W_{n+3}+st(r^2+s)W_{n+2}W_{n+1})$

$+(n+2)(n+3)z^{n+1}((r^2+s)W_{n+3}^2-s(r^2+s)W_{n+2}^2+(t-r^3)W_{n+2}W_{n+3}-s(r^2+s)W_{n+3}W_{n+1}-t(r^2+s)W_{n+2}W_{n+1})+(n+1)(n+2)z^n(sW_{n+2}^2+rW_{n+2}W_{n+3}-(r^2+s)W_{n+1}W_{n+3}+tW_{n+1}W_{n+2})+n(n+1)z^{n-1}W_{n+1}W_{n+3}+20z^3t^3(W_2-rW_1-sW_0)W_1+12z^2t(r(s^2-rt)W_1^2+tW_0^2(s^2-rt)+(rt-s^2)W_1W_2-stW_0W_2+(s^3+t^2)W_0W_1)$

$+6z((s^2-rt)W_2^2-t^2(r^2+s)W_0^2+r\left(rt-s^2\right)W_1W_2+(t^2-s^3+2rst)W_0W_2-st(r^2+s)W_0W_1)+2(-(r^2+s)W_2^2+s(r^2+s)W_1^2+(r^3-t)W_1W_2+s(r^2+s)W_0W_2+t(r^2+s)W_0W_1)$

and
$\frac{d^2}{dz^2}\Gamma(z) = -30z^4t^4 + 20z^3t^2(s^2 - rt) + 12z^2t(r^2t - rs^2 - st) + 6z(r^3t - s^3 + 2t^2 + 4rst) + 2(r^2s + s^2 + rt)$

(iv): *If* $\Gamma(z) = (-t^2z^3 + sz + rtz^2 + 1)(r^2z - s^2z^2 + t^2z^3 + 2sz + 2rtz^2 - 1) = (z - a_1)^3 f(z) = 0$ *for some* $a_1 \in \mathbb{C}$ *and a function* f *in* z *with* $f(a_1) \neq 0$ *then, for* $z = a_1$, *we get*

$$\sum_{k=0}^{n} z^k W_{k+2} W_k = \frac{\frac{d^3}{dz^3}\Theta_{3W}(z)}{\frac{d^3}{dz^3}\Gamma(z)} = \frac{\frac{d^3}{dz^3}\Theta_{3W}(z)}{-120z^3t^4 + 60z^2t^2(s^2 - rt) + 24zt(r^2t - rs^2 - st) + 6(r^3t - s^3 + 2t^2 + 4rst)} \tag{2.18}$$

where
$\frac{d^3}{dz^3}\Theta_{3W}(z) = (n+4)(n+5)(n+6)z^{n+3}t^3(-W_{n+3} + rW_{n+2} + sW_{n+1})W_{n+2} + (n+3)(n+4)(n+5)z^{n+2}t(r(rt - s^2)W_{n+2}^2 + t(rt - s^2)W_{n+1}^2 + (s^2 - rt)W_{n+2}W_{n+3} - (s^3 + t^2)W_{n+1}W_{n+2} + stW_{n+1}W_{n+3}) + (n+2)(n+3)(n+4)z^{n+1}((rt - s^2)W_{n+3}^2 + t^2(r^2 + s)W_{n+1}^2 + r(s^2 - rt)W_{n+2}W_{n+3} + (s^3 - 2rst - t^2)W_{n+1}W_{n+3} + st(r^2 + s)W_{n+2}W_{n+1})$
$+(n+1)(n+2)(n+3)z^n((r^2 + s)W_{n+3}^2 - s(r^2 + s)W_{n+2}^2 + (t - r^3)W_{n+2}W_{n+3} - s(r^2 + s)W_{n+3}W_{n+1} - t(r^2 + s)W_{n+2}W_{n+1}) + n(n+1)(n+2)z^{n-1}(sW_{n+2}^2 + rW_{n+2}W_{n+3} - (r^2 + s)W_{n+1}W_{n+3} + tW_{n+1}W_{n+2}) + (n-1)n(n+1)z^{n-2}W_{n+1}W_{n+3}$
$+60z^2t^3(W_2 - rW_1 - sW_0)W_1 + 24zt(r(s^2 - rt)W_1^2 + tW_0^2(s^2 - rt) + (rt - s^2)W_1W_2 - stW_0W_2 + (s^3 + t^2)W_0W_1) + 6((s^2 - rt)W_2^2 - t^2(r^2 + s)W_0^2 + r\left(rt - s^2\right)W_1W_2 + (t^2 - s^3 + 2rst)W_0W_2 - st(r^2 + s)W_0W_1)$

and
$\frac{d^3}{dz^3}\Gamma(z) = -120z^3t^4 + 60z^2t^2(s^2 - rt) + 24zt(r^2t - rs^2 - st) + 6(r^3t - s^3 + 2t^2 + 4rst)$

(v): *If* $\Gamma(z) = (-t^2z^3 + sz + rtz^2 + 1)(r^2z - s^2z^2 + t^2z^3 + 2sz + 2rtz^2 - 1) = (z - a_1)^4 f(z) = 0$ *for some* $a_1 \in \mathbb{C}$ *and a function* f *in* z *with* $f(a_1) \neq 0$ *then, for* $z = a_1$, *we get*

$$\sum_{k=0}^{n} z^k W_{k+2} W_k = \frac{\frac{d^4}{dz^4}\Theta_{3W}(z)}{\frac{d^4}{dz^4}\Gamma(z)} = \frac{\frac{d^4}{dz^4}\Theta_{3W}(z)}{-360z^2t^4 + 120zt^2(s^2 - rt) + 24t(r^2t - rs^2 - st)} \tag{2.19}$$

where
$\frac{d^4}{dz^4}\Theta_{3W}(z) = (n+3)(n+4)(n+5)(n+6)z^{n+2}t^3(-W_{n+3} + rW_{n+2} + sW_{n+1})W_{n+2} + (n+2)(n+3)(n+4)(n+5)z^{n+1}t(r(rt - s^2)W_{n+2}^2 + t(rt - s^2)W_{n+1}^2 + (s^2 - rt)W_{n+2}W_{n+3} - (s^3 + t^2)W_{n+1}W_{n+2} + stW_{n+1}W_{n+3})$

$+(n+1)(n+2)(n+3)(n+4)z^{n}((rt-s^{2})W_{n+3}^{2}+t^{2}(r^{2}+s)W_{n+1}^{2}+r(s^{2}-rt)W_{n+2}W_{n+3}+(s^{3}-2rst-t^{2})W_{n+1}W_{n+3}+st(r^{2}+s)W_{n+2}W_{n+1})+n(n+1)(n+2)(n+3)z^{n-1}((r^{2}+s)W_{n+3}^{2}-s(r^{2}+s)W_{n+2}^{2}+(t-r^{3})W_{n+2}W_{n+3}-s(r^{2}+s)W_{n+3}W_{n+1}-t(r^{2}+s)W_{n+2}W_{n+1})$
$+(n-1)n(n+1)(n+2)z^{n-2}(sW_{n+2}^{2}+rW_{n+2}W_{n+3}-(r^{2}+s)W_{n+1}W_{n+3}+tW_{n+1}W_{n+2})+(n-2)(n-1)n(n+1)z^{n-3}W_{n+1}W_{n+3}+120zt^{3}(W_{2}-rW_{1}-sW_{0})W_{1}+24t(r(s^{2}-rt)W_{1}^{2}+tW_{0}^{2}(s^{2}-rt)+(rt-s^{2})W_{1}W_{2}-stW_{0}W_{2}+(s^{3}+t^{2})W_{0}W_{1})$

and

$\frac{d^{4}}{dz^{4}}\Gamma(z)=-360z^{2}t^{4}+120zt^{2}(s^{2}-rt)+24t(r^{2}t-rs^{2}-st)$

(vi): *If* $\Gamma(z)=(-t^{2}z^{3}+sz+rtz^{2}+1)(r^{2}z-s^{2}z^{2}+t^{2}z^{3}+2sz+2rtz^{2}-1)=(z-a_{1})^{5}f(z)=0$ *for some* $a_{1}\in\mathbb{C}$ *and a function* f *in* z *with* $f(a_{1})\neq 0$ *then, for* $z=a_{1}$, *we get*

$$\sum_{k=0}^{n}z^{k}W_{k+2}W_{k} = \frac{\frac{d^{5}}{dz^{5}}\Theta_{3W}(z)}{\frac{d^{5}}{dz^{5}}\Gamma(z)} = \frac{\frac{d^{5}}{dz^{5}}\Theta_{3W}(z)}{-720zt^{4}+120t^{2}(s^{2}-rt)} \tag{2.20}$$

where

$\frac{d^{5}}{dz^{5}}\Theta_{3W}(z)=(n+2)(n+3)(n+4)(n+5)(n+6)z^{n+1}t^{3}(-W_{n+3}+rW_{n+2}+sW_{n+1})W_{n+2}+(n+1)(n+2)(n+3)(n+4)(n+5)z^{n}t(r(rt-s^{2})W_{n+2}^{2}+t(rt-s^{2})W_{n+1}^{2}+(s^{2}-rt)W_{n+2}W_{n+3}-(s^{3}+t^{2})W_{n+1}W_{n+2}+stW_{n+1}W_{n+3})$
$+n(n+1)(n+2)(n+3)(n+4)z^{n-1}((rt-s^{2})W_{n+3}^{2}+t^{2}(r^{2}+s)W_{n+1}^{2}+r(s^{2}-rt)W_{n+2}W_{n+3}+(s^{3}-2rst-t^{2})W_{n+1}W_{n+3}+st(r^{2}+s)W_{n+2}W_{n+1})+(n-1)n(n+1)(n+2)(n+3)z^{n-2}((r^{2}+s)W_{n+3}^{2}-s(r^{2}+s)W_{n+2}^{2}+(t-r^{3})W_{n+2}W_{n+3}-s(r^{2}+s)W_{n+3}W_{n+1}-t(r^{2}+s)W_{n+2}W_{n+1})$
$+(n-2)(n-1)n(n+1)(n+2)z^{n-3}(sW_{n+2}^{2}+rW_{n+2}W_{n+3}-(r^{2}+s)W_{n+1}W_{n+3}+tW_{n+1}W_{n+2})+(n-3)(n-2)(n-1)n(n+1)z^{n-4}W_{n+1}W_{n+3}+120t^{3}(W_{2}-rW_{1}-sW_{0})W_{1}$

and

$\frac{d^{5}}{dz^{5}}\Gamma(z)=-720zt^{4}+120t^{2}(s^{2}-rt)$

(vii): *If* $\Gamma(z)=(-t^{2}z^{3}+sz+rtz^{2}+1)(r^{2}z-s^{2}z^{2}+t^{2}z^{3}+2sz+2rtz^{2}-1)=(z-a_{1})^{6}=0$ *for some* $a_{1}\in\mathbb{C}$ *then, for* $z=a_{1}$, *we get*

$$\sum_{k=0}^{n}z^{k}W_{k+2}W_{k} = \frac{\frac{d^{6}}{dz^{6}}\Theta_{3W}(z)}{\frac{d^{6}}{dz^{6}}\Gamma(z)} = \frac{\frac{d^{6}}{dz^{6}}\Theta_{3W}(z)}{-720t^{4}} \tag{2.21}$$

where

$\frac{d^6}{dz^6}\Theta_{3W}(z) = (n+1)(n+2)(n+3)(n+4)(n+5)(n+6)z^n t^3(-W_{n+3}+rW_{n+2}+sW_{n+1})W_{n+2}+$
$n(n+1)(n+2)(n+3)(n+4)(n+5)z^{n-1}t(r(rt-s^2)W_{n+2}^2+t(rt-s^2)W_{n+1}^2+(s^2-rt)W_{n+2}W_{n+3}-$
$(s^3+t^2)W_{n+1}W_{n+2}+stW_{n+1}W_{n+3})$
$+(n-1)n(n+1)(n+2)(n+3)(n+4)z^{n-2}((rt-s^2)W_{n+3}^2+t^2(r^2+s)W_{n+1}^2+r(s^2-rt)W_{n+2}W_{n+3}+$
$(s^3-2rst-t^2)W_{n+1}W_{n+3}+st(r^2+s)W_{n+2}W_{n+1})+(n-2)(n-1)n(n+1)(n+2)(n+3)z^{n-3}((r^2+$
$s)W_{n+3}^2-s(r^2+s)W_{n+2}^2+(t-r^3)W_{n+2}W_{n+3}-s(r^2+s)W_{n+3}W_{n+1}-t(r^2+s)W_{n+2}W_{n+1})$
$+(n-3)(n-2)(n-1)n(n+1)(n+2)z^{n-4}(sW_{n+2}^2+rW_{n+2}W_{n+3}-(r^2+s)W_{n+1}W_{n+3}+$
$tW_{n+1}W_{n+2})+(n-4)(n-3)(n-2)(n-1)n(n+1)z^{n-5}W_{n+1}W_{n+3}$
and
$\frac{d^6}{dz^6}\Gamma(z) = -720t^4$

Proof.

(a)(i), (b)(i), (c)(i). First, we obtain $\sum_{k=0}^{n} W_k^2$. Using the recurrence relation

$$W_{n+3} = rW_{n+2} + sW_{n+1} + tW_n$$

or

$$tW_n = W_{n+3} - rW_{n+2} - sW_{n+1}$$

i.e.

$$t^2W_n^2 = (W_{n+3}-rW_{n+2}-sW_{n+1})^2 = W_{n+3}^2+r^2W_{n+2}^2+s^2W_{n+1}^2-2rW_{n+3}W_{n+2}-2sW_{n+3}W_{n+1}+2rsW_{n+2}W_{n+1}$$

we obtain

$$\begin{aligned}
t^2z^nW_n^2 &= z^nW_{n+3}^2+r^2z^nW_{n+2}^2+s^2z^nW_{n+1}^2-2rz^nW_{n+3}W_{n+2}-2sz^nW_{n+3}W_{n+1}+2rsz^nW_{n+2}W_{n+1}\\
t^2z^{n-1}W_{n-1}^2 &= z^{n-1}W_{n+2}^2+r^2z^{n-1}W_{n+1}^2+s^2z^{n-1}W_n^2-2rz^{n-1}W_{n+2}W_{n+1}\\
&\quad -2sz^{n-1}W_{n+2}W_n+2rsz^{n-1}W_{n+1}W_n\\
t^2z^{n-2}W_{n-2}^2 &= z^{n-2}W_{n+1}^2+r^2z^{n-2}W_n^2+s^2z^{n-2}W_{n-1}^2-2rz^{n-2}W_{n+1}W_n\\
&\quad -2sz^{n-2}W_{n+1}W_{n-1}+2rsz^{n-2}W_nW_{n-1}\\
&\quad \vdots\\
t^2z^2W_2^2 &= z^2W_5^2+r^2z^2W_4^2+s^2z^2W_3^2-2rz^2W_5W_4-2sz^2W_5W_3+2rsz^2W_4W_3\\
t^2z^1W_1^2 &= z^1W_4^2+r^2z^1W_3^2+s^2z^1W_2^2-2rz^1W_4W_3-2sz^1W_4W_2+2rsz^1W_3W_2\\
t^2z^0W_0^2 &= z^0W_3^2+r^2z^0W_2^2+s^2z^0W_1^2-2rz^0W_3W_2-2sz^0W_3W_1+2rsz^0W_2W_1
\end{aligned}$$

If we add the equations side by side, we get

$$\begin{aligned} t^2\sum_{k=0}^{n} z^k W_k^2 &= \sum_{k=3}^{n+3} z^{k-3}W_k^2 + r^2\sum_{k=2}^{n+2} z^{k-2}W_k^2 + s^2\sum_{k=1}^{n+1} z^{k-1}W_k^2 \\ &\quad -2r\sum_{k=2}^{n+2} z^{k-2}W_{k+1}W_k - 2s\sum_{k=1}^{n+1} z^{k-1}W_{k+2}W_k + 2rs\sum_{k=1}^{n+1} z^{k-1}W_{k+1}W_k \end{aligned} \tag{2.22}$$

Next we obtain $\sum_{k=0}^{n} W_{k+1}W_k$. Multiplying the both side of the recurrence relation

$$tW_n = W_{n+3} - rW_{n+2} - sW_{n+1}$$

by W_{n+1} we get

$$tW_{n+1}W_n = W_{n+3}W_{n+1} - rW_{n+2}W_{n+1} - sW_{n+1}^2.$$

Then using last recurrence relation, we obtain

$$\begin{aligned} tz^nW_{n+1}W_n &= z^nW_{n+3}W_{n+1} - rz^nW_{n+2}W_{n+1} - sz^nW_{n+1}^2 \\ tz^{n-1}W_nW_{n-1} &= z^{n-1}W_{n+2}W_n - rz^{n-1}W_{n+1}W_n - sz^{n-1}W_n^2 \\ tz^{n-2}W_{n-1}W_{n-2} &= z^{n-2}W_{n+1}W_{n-1} - rz^{n-2}W_nW_{n-1} - sz^{n-2}W_{n-1}^2 \\ &\vdots \\ tz^2W_3W_2 &= z^2W_5W_3 - rz^2W_4W_3 - sz^2W_3^2 \\ tzW_2W_1 &= zW_4W_2 - rzW_3W_2 - szW_2^2 \\ tz^0W_1W_0 &= z^0W_3W_1 - rz^0W_2W_1 - sz^0W_1^2 \end{aligned}$$

If we add the equations side by side, we get

$$t\sum_{k=0}^{n} z^kW_{k+1}W_k = \sum_{k=1}^{n+1} z^{k-1}W_{k+2}W_k - r\sum_{k=1}^{n+1} z^{k-1}W_{k+1}W_k - s\sum_{k=1}^{n+1} z^{k-1}W_k^2. \tag{2.23}$$

Next we obtain $\sum_{k=2}^{n} W_{k+2}W_k$. Multiplying the both side of the recurrence relation

$$tW_n = W_{n+3} - rW_{n+2} - sW_{n+1}$$

by W_{n+2} we get

$$tW_{n+2}W_n = W_{n+3}W_{n+2} - rW_{n+2}^2 - sW_{n+2}W_{n+1}.$$

Then using last recurrence relation, we obtain

$$
\begin{aligned}
tz^nW_{n+2}W_n &= z^nW_{n+3}W_{n+2} - rz^nW_{n+2}^2 - sz^nW_{n+2}W_{n+1} \\
tz^{n-1}W_{n+1}W_{n-1} &= z^{n-1}W_{n+2}W_{n+1} - rz^{n-1}W_{n+1}^2 - sz^{n-1}W_{n+1}W_n \\
tz^{n-2}W_nW_{n-2} &= z^{n-2}W_{n+1}W_n - rz^{n-2}W_n^2 - sz^{n-2}W_nW_{n-1} \\
&\vdots \\
tz^2W_4W_2 &= z^2W_5W_4 - rz^2W_4^2 - sz^2W_4W_3 \\
tz^1W_3W_1 &= z^1W_4W_3 - rz^1W_3^2 - sz^1W_3W_2 \\
tz^0W_2W_0 &= z^0W_3W_2 - rz^0W_2^2 - sz^0W_2W_1
\end{aligned}
$$

If we add the equations side by side, we get

$$
t\sum_{k=0}^{n} z^kW_{k+2}W_k = \sum_{k=2}^{n+2} z^{k-2}W_{k+1}W_k - r\sum_{k=2}^{n+2} z^{k-2}W_k^2 - s\sum_{k=1}^{n+1} z^{k-1}W_{k+1}W_k \tag{2.24}
$$

Solving the system (2.22)-(2.23)-(2.24), the results in (a)(i), (b)(i), (c)(i) follow.

(a):

(ii): We use (2.1). For $z = a_1$, the right hand side of the sum formula (2.1) is an indeterminate form. Now, we can use L'Hospital rule. Then we get (ii) by using

$$
\sum_{k=0}^{n} a_1^kW_k^2 = \left.\frac{\frac{d}{dz}\Theta_{1W}(z)}{\frac{d}{dz}\Gamma(z)}\right|_{z=a_1}.
$$

(iii): For $z = a_1$, the right hand side of the sum formula (2.1) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (iii) by using

$$
\sum_{k=0}^{n} a_1^kW_k^2 = \left.\frac{\frac{d^2}{dz^2}\Theta_{1W}(z)}{\frac{d^2}{dz^2}\Gamma(z)}\right|_{z=a_1}.
$$

(iv): For $z = a_1$, the right hand side of the sum formula (2.1) is an indeterminate form. Now, we can use L'Hospital rule (three times). Then we get (iv) by using

$$
\sum_{k=0}^{n} a_1^kW_k^2 = \left.\frac{\frac{d^3}{dz^3}\Theta_{1W}(z)}{\frac{d^3}{dz^3}\Gamma(z)}\right|_{z=a_1}.
$$

(v): For $z = a_1$, the right hand side of the sum formula (2.1) is an indeterminate form. Now, we can use L'Hospital rule (four times). Then we get (v) by using

$$
\sum_{k=0}^{n} a_1^kW_k^2 = \left.\frac{\frac{d^4}{dz^4}\Theta_{1W}(z)}{\frac{d^4}{dz^4}\Gamma(z)}\right|_{z=a_1}.
$$

(vi): For $z = a_1$, the right hand side of the sum formula (2.1) is an indeterminate form. Now, we can use L'Hospital rule (five times). Then we get (vi) by using

$$\sum_{k=0}^{n} a_1^k W_k^2 = \left. \frac{\frac{d^5}{dz^5}\Theta_{1W}(z)}{\frac{d^5}{dz^5}\Gamma(z)} \right|_{z=a_1}.$$

(vii): For $z = a_1$, the right hand side of the sum formula (2.1) is an indeterminate form. Now, we can use L'Hospital rule (six times). Then we get (vii) by using

$$\sum_{k=0}^{n} a_1^k W_k^2 = \left. \frac{\frac{d^6}{dz^6}\Theta_{1W}(z)}{\frac{d^6}{dz^6}\Gamma(z)} \right|_{z=a_1}.$$

(b):

(ii): We use (2.8). For $z = a_1$, the right hand side of the sum formula (2.8) is an indeterminate form. Now, we can use L'Hospital rule. Then we get (ii) by using

$$\sum_{k=0}^{n} a_1^k W_{k+1} W_k = \left. \frac{\frac{d}{dz}\Theta_{2W}(z)}{\frac{d}{dz}\Gamma(z)} \right|_{z=a_1}.$$

(iii): For $z = a_1$, the right hand side of the sum formula (2.8) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (iii) by using

$$\sum_{k=0}^{n} a_1^k W_{k+1} W_k = \left. \frac{\frac{d^2}{dz^2}\Theta_{2W}(z)}{\frac{d^2}{dz^2}\Gamma(z)} \right|_{z=a_1}.$$

(iv): For $z = a_1$, the right hand side of the sum formula (2.8) is an indeterminate form. Now, we can use L'Hospital rule (three times). Then we get (iv) by using

$$\sum_{k=0}^{n} a_1^k W_{k+1} W_k = \left. \frac{\frac{d^3}{dz^3}\Theta_{2W}(z)}{\frac{d^3}{dz^3}\Gamma(z)} \right|_{z=a_1}.$$

(v): For $z = a_1$, the right hand side of the sum formula (2.8) is an indeterminate form. Now, we can use L'Hospital rule (four times). Then we get (v) by using

$$\sum_{k=0}^{n} a_1^k W_{k+1} W_k = \left. \frac{\frac{d^4}{dz^4}\Theta_{2W}(z)}{\frac{d^4}{dz^4}\Gamma(z)} \right|_{z=a_1}.$$

(vi): For $z = a_1$, the right hand side of the sum formula (2.8) is an indeterminate form. Now, we can use L'Hospital rule (five times). Then we get (vi) by using

$$\sum_{k=0}^{n} a_1^k W_{k+1} W_k = \left. \frac{\frac{d^5}{dz^5}\Theta_{2W}(z)}{\frac{d^5}{dz^5}\Gamma(z)} \right|_{z=a_1}.$$

(vii): For $z = a_1$, the right hand side of the sum formula (2.8) is an indeterminate form. Now, we can use L'Hospital rule (six times). Then we get (vii) by using

$$\sum_{k=0}^{n} a_1^k W_{k+1} W_k = \left. \frac{\frac{d^6}{dz^6}\Theta_{2W}(z)}{\frac{d^6}{dz^6}\Gamma(z)} \right|_{z=a_1}.$$

(c):

(ii): We use (2.15). For $z = a_1$, the right hand side of the sum formula (2.15) is an indeterminate form. Now, we can use L'Hospital rule. Then we get (ii) by using

$$\sum_{k=0}^{n} a_1^k W_{k+2} W_k = \left. \frac{\frac{d}{dz}\Theta_{3W}(z)}{\frac{d}{dz}\Gamma(z)} \right|_{z=a_1}.$$

(iii): For $z = a_1$, the right hand side of the sum formula (2.15) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (iii) by using

$$\sum_{k=0}^{n} a_1^k W_{k+2} W_k = \left. \frac{\frac{d^2}{dz^2}\Theta_{3W}(z)}{\frac{d^2}{dz^2}\Gamma(z)} \right|_{z=a_1}.$$

(iv): For $z = a_1$, the right hand side of the sum formula (2.15) is an indeterminate form. Now, we can use L'Hospital rule (three times). Then we get (iv) by using

$$\sum_{k=0}^{n} a_1^k W_{k+2} W_k = \left. \frac{\frac{d^3}{dz^3}\Theta_{3W}(z)}{\frac{d^3}{dz^3}\Gamma(z)} \right|_{z=a_1}.$$

(v): For $z = a_1$, the right hand side of the sum formula (2.15) is an indeterminate form. Now, we can use L'Hospital rule (four times). Then we get (v) by using

$$\sum_{k=0}^{n} a_1^k W_{k+2} W_k = \left. \frac{\frac{d^4}{dz^4}\Theta_{3W}(z)}{\frac{d^4}{dz^4}\Gamma(z)} \right|_{z=a_1}.$$

(vi): For $z = a_1$, the right hand side of the sum formula (2.15) is an indeterminate form. Now, we can use L'Hospital rule (five times). Then we get (vi) by using

$$\sum_{k=0}^{n} a_1^k W_{k+2} W_k = \left. \frac{\frac{d^5}{dz^5}\Theta_{3W}(z)}{\frac{d^5}{dz^5}\Gamma(z)} \right|_{z=a_1}.$$

(vii): For $z = a_1$, the right hand side of the sum formula (2.15) is an indeterminate form. Now, we can use L'Hospital rule (six times). Then we get (vii) by using

$$\sum_{k=0}^{n} a_1^k W_{k+2} W_k = \left. \frac{\frac{d^6}{dz^6}\Theta_{3W}(z)}{\frac{d^6}{dz^6}\Gamma(z)} \right|_{z=a_1}. \qquad \square$$

Remark 2.2. *According to roots of* $\Gamma(z) = (-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1) = 0$, *the sum formulas* $\sum_{k=0}^{n} z^k W_k^2$, $\sum_{k=0}^{n} z^k W_{k+1}W_k$ *and* $\sum_{k=0}^{n} z^k W_{k+2}W_k$ *can be evaluated by using Theorem 2.1. For example,*

- *If* $\Gamma(z) = (-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1) = u(z-a_1)(z-a_2)(z-a_3)(z-a_4)(z-a_5)(z-a_6) = 0$ *for some* $u, a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{C}$ *with* $u \neq 0$ *and* $a_1 \neq a_2 \neq a_3 \neq a_4 \neq a_5 \neq a_6$, *i.e.,* $z = a_1$ *or* $z = a_2$ *or* $z = a_3$ *or* $z = a_4$ *or* $z = a_5$ *or* $z = a_6$ *then we use (2.2) in (a)(ii), (2.9) in (b)(ii) and (2.16) in (c)(ii) to calculate* $\sum_{k=0}^{n} z^k W_k^2$, $\sum_{k=0}^{n} z^k W_{k+1}W_k$ *and* $\sum_{k=0}^{n} z^k W_{k+2}W_k$, *respectively.*
- *If* $\Gamma(z) = (-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1) = u(z-a_1)^3(z-a_2)^2(z-a_3) = 0$ *for some* $u, a_1, a_2, a_3 \in \mathbb{C}$ *with* $u \neq 0$ *and* $a_1 \neq a_2 \neq a_3$, *i.e.,* $z = a_1$ *or* $z = a_2$ *or* $z = a_3$

  then
  - *if* $z = a_1$ *then we use (2.4) in (a)(iv), (2.11) in (b)(iv) and (2.18) in (c)(iv) to calculate* $\sum_{k=0}^{n} z^k W_k^2$, $\sum_{k=0}^{n} z^k W_{k+1}W_k$ *and* $\sum_{k=0}^{n} z^k W_{k+2}W_k$, *respectively,*
  - *if* $z = a_2$ *then we use (2.3) in (a)(iii), (2.10) in (b)(iii) and (2.17) in (c)(iii) to calculate* $\sum_{k=0}^{n} z^k W_k^2$, $\sum_{k=0}^{n} z^k W_{k+1}W_k$ *and* $\sum_{k=0}^{n} z^k W_{k+2}W_k$, *respectively,*
  - *if* $z = a_3$ *then we use (2.2) in (a)(ii), (2.9) in (b)(ii) and (2.16) in (c)(ii) to calculate* $\sum_{k=0}^{n} z^k W_k^2$, $\sum_{k=0}^{n} z^k W_{k+1}W_k$ *and* $\sum_{k=0}^{n} z^k W_{k+2}W_k$, *respectively.*
- *If* $\Gamma(z) = (-t^2z^3 + sz + rtz^2 + 1)(r^2z - s^2z^2 + t^2z^3 + 2sz + 2rtz^2 - 1) = u(z-a_1)^4(z-a_2)^2 = 0$ *for some* $u, a_1, a_2 \in \mathbb{C}$ *with* $u \neq 0$ *and* $a_1 \neq a_2$, *i.e.,* $z = a_1$ *or* $z = a_2$

  then
  - *if* $z = a_1$ *then we use (2.5) in (a)(v), (2.12) in (b)(v) and (2.19) in (c)(v) to calculate* $\sum_{k=0}^{n} z^k W_k^2$, $\sum_{k=0}^{n} z^k W_{k+1}W_k$ *and* $\sum_{k=0}^{n} z^k W_{k+2}W_k$, *respectively,*
  - *if* $z = a_2$ *then we use (2.3) in (a)(iii), (2.10) in (b)(iii) and (2.17) in (c)(iii) to calculate* $\sum_{k=0}^{n} z^k W_k^2$, $\sum_{k=0}^{n} z^k W_{k+1}W_k$ *and* $\sum_{k=0}^{n} z^k W_{k+2}W_k$, *respectively,*

3. Generating Functions

In this section, we present the closed forms of formulas of generating functions $\sum_{n=0}^{\infty} W_n^2 z^n$, $\sum_{n=0}^{\infty} W_{n+1}W_n z^n$ and $\sum_{n=0}^{\infty} W_{n+2}W_n z^n$ for the generalized Tribonacci polynomials.

Theorem 3.1. *Assume that* $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\alpha\beta|^{-1}, |\alpha\gamma|^{-1}, |\beta\gamma|^{-1}\}$. *Then*

(a): *The ordinary generating function* $\sum_{n=0}^{\infty} W_n^2 z^n$ *of the sequence* $\{W_n^2\}$ *is given by*

$$\sum_{n=0}^{\infty} W_n^2 z^n = \frac{\Psi_1(z)}{(-t^2z^3 + sz + rtz^2 + 1)(r^2z - s^2z^2 + t^2z^3 + 2sz + 2rtz^2 - 1)}$$

where

$$\Psi_1(z) = z^5\Theta_7 + z^4\Theta_8 + z^3\Theta_9 + z^2\Theta_{10} + z\Theta_{11} + \Theta_{12}$$

$= z^5t^2(-W_2 + rW_1 + sW_0)^2 + z^4t(rW_2^2 + (t + 2rs + r^3)W_1^2 + r(rt - s^2)W_0^2 - 2(s + r^2)W_1W_2 - 2(rt - s^2)W_0W_1) + z^3(sW_2^2 + r(t + rs)W_1^2 + (r^3t - s^3 + t^2 + 4rst)W_0^2 - 2rsW_1W_2 - 2rtW_0W_2 - 2stW_0W_1) + z^2(-W_2^2 + (r^2 + s)W_1^2 + s(s + r^2)W_0^2 + rtW_0^2) + z(-W_1^2 + (r^2 + s)W_0^2) - W_0^2$

(b): *The ordinary generating function* $\sum_{n=0}^{\infty} W_{n+1}W_nz^n$ *of the sequence* $\{W_{n+1}W_n\}$ *is given by*

$$\sum_{n=0}^{\infty} W_{n+1}W_nz^n = \frac{\Psi_2(z)}{(-t^2z^3 + sz + rtz^2 + 1)(r^2z - s^2z^2 + t^2z^3 + 2sz + 2rtz^2 - 1)}$$

where

$\Psi_2(z) = z^5\Theta_{19} + z^4\Theta_{20} + z^3\Theta_{21} + z^2\Theta_{22} + z\Theta_{23} + \Theta_{24}$

$= z^5t^3(W_2 - rW_1 - sW_0)W_0 + z^4t(W_2 - rW_1 - sW_0)(-sW_2 + (rs + t)W_1) + z^3(-s(t + rs)W_1^2 - rt^2W_0^2 + s^2W_1W_2 - r^2tW_0W_2 + (r^3t - s^3 + t^2 + 2rst)W_0W_1) + z^2(-rW_2^2 + r^2W_1W_2 - tW_0W_2 + (r^2s + rt + s^2)W_0W_1) + z(-W_2 + (r^2 + s)W_0)W_1 - W_0W_1$

(c): *The ordinary generating function* $\sum_{n=0}^{\infty} W_{n+2}W_nz^n$ *of the sequence* $\{W_{n+2}W_n\}$ *is given by*

$$\sum_{n=0}^{\infty} W_{n+2}W_nz^n = \frac{\Psi_3(z)}{(-t^2z^3 + sz + rtz^2 + 1)(r^2z - s^2z^2 + t^2z^3 + 2sz + 2rtz^2 - 1)}$$

where

$\Psi_3(z) = z^5\Theta_{31} + z^4\Theta_{32} + z^3\Theta_{33} + z^2\Theta_{34} + z\Theta_{35} + \Theta_{36}$

$= z^5t^3(W_2 - rW_1 - sW_0)W_1 + z^4t(r(s^2 - rt)W_1^2 + tW_0^2(s^2 - rt) + (rt - s^2)W_1W_2 - stW_0W_2 + (s^3 + t^2)W_0W_1) + z^3((s^2 - rt)W_2^2 - t^2(r^2 + s)W_0^2 + r\left(rt - s^2\right)W_1W_2 + (t^2 - s^3 + 2rst)W_0W_2 - st(r^2 + s)W_0W_1) + z^2(-(r^2 + s)W_2^2 + s(r^2 + s)W_1^2 + (r^3 - t)W_1W_2 + s(r^2 + s)W_0W_2 + t(r^2 + s)W_0W_1) + z(-sW_1^2 - rW_1W_2 + (r^2 + s)W_0W_2 - tW_0W_1) - W_0W_2$

Proof. Use Theorem 2.1 (a)(i), (b)(i), (c)(i) and Theorem 1.2. □

References

[1] Cerda-Moralez, G., On Third-Order Jacobsthal Polynomials and Their Properties, Miskolc Mathematical Notes, Vol. 22(1), 123–132, 2021. DOI: 10.18514/MMN.2021.3227

[2] Merzouka, H., Boussayoudb, A., Chelghamc, M., Generating Functions of Generalized Tribonacci and Tricobsthal Polynomials, Montes Taurus Journal of Pure and Applied Mathematics, 2(2), 7–37, 2020.

[3] Özkan, E., Altun, İ., (2019) Generalized Lucas polynomials and relationships between the Fibonacci polynomials and Lucas polynomials, Communications in Algebra, 47(10), 2019. DOI: 10.1080/00927872.2019.1576186

[4] Ricci, P.E., A Note on Q-matrices and Higher Order Fibonacci Polynomials, Notes on Number Theory and Discrete Mathematics, 27(1), 2021, 91-100. DOI: 10.7546/nntdm.2021.27.1.91-100

[5] Soykan, Y., Generalized Tribonacci Polynomials, Earthline Journal of Mathematical Sciences, 13(1), 1-120, 2023. https://doi.org/10.34198/ejms.13123.1120